Supplementary Online Appendix
for "Protection for Sale: The Case of Oligopolistic Competition and Interdependent Sectors"

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This appendix provides detailed derivations of some results presented in the main text.

B.1 Derivation of equation (6)

To derive equation (6), substitute the expressions for tariff revenue and for lobby i's welfare into (5). This yields

$$(a+\alpha_L)\nabla\left[\sum_{k=1}^{m}\tau_k n_k^* q_k^*(\boldsymbol{\tau})-\sum_{k=1}^{m}p_k d_k(\mathbf{p}(\boldsymbol{\tau}))+\hat{U}(d_1(\mathbf{p}(\boldsymbol{\tau})),..,d_m(\mathbf{p}(\boldsymbol{\tau})))\right] + \sum_{k=1}^{m}(I_k+a)n_k\nabla\pi_k(\boldsymbol{\tau}) = \mathbf{0}. \quad \text{(B1)}$$

As

$$\frac{\partial}{\partial\tau_j}\left(\sum_{k=1}^{m}\tau_k n_k^* q_k^*(\boldsymbol{\tau})\right)=n_j^* q_j^*(\boldsymbol{\tau})+\sum_{k=1}^{m}\tau_k n_k^*\frac{\partial q_k^*(\boldsymbol{\tau})}{\partial\tau_j},$$

$$\frac{\partial}{\partial\tau_j}\left(-\sum_{k=1}^{m}p_k d_k(\mathbf{p}(\boldsymbol{\tau}))\right)=\left(-\sum_{k=1}^{m}\frac{\partial p_k}{\partial\tau_j}d_k(\mathbf{p}(\boldsymbol{\tau}))-\sum_{k=1}^{m}p_k\frac{\partial d_k(\mathbf{p}(\boldsymbol{\tau}))}{\partial\tau_j}\right),$$

and

$$\frac{\partial}{\partial\tau_j}\left(\hat{U}(d_1(\mathbf{p}(\boldsymbol{\tau})),...,d_m(\mathbf{p}(\boldsymbol{\tau})))\right)=\sum_{i=1}^{m}\frac{\partial\hat{U}(d_1(\mathbf{p}),...,d_m(\mathbf{p}))}{\partial\left(d_i(\mathbf{p})\right)}\frac{\partial d_i(\mathbf{p}(\boldsymbol{\tau}))}{\partial\tau_j}=\sum_{i=1}^{m}p_i\frac{\partial d_i(\mathbf{p}(\boldsymbol{\tau}))}{\partial\tau_j},$$

each line of condition (B1) takes the form

$$(a+\alpha_L)\left(n_j^* q_j^*(\boldsymbol{\tau})+\sum_{k=1}^{m}\tau_k n_k^*\frac{\partial q_k^*(\boldsymbol{\tau})}{\partial\tau_j}-\sum_{k=1}^{m}\frac{\partial p_k}{\partial\tau_j}d_k(\mathbf{p}(\boldsymbol{\tau}))\right)+\sum_{k=1}^{m}(I_k+a)n_k\frac{\partial\pi_k(\boldsymbol{\tau})}{\partial\tau_j}=\mathbf{0},$$

which can be rewritten as

$$-\sum_{k=1}^{m}\tau_k n_k^*\frac{\partial q_k^*(\boldsymbol{\tau})}{\partial\tau_j}=\sum_{k=1}^{m}\frac{(I_k+a)}{(a+\alpha_L)}n_k\frac{\partial\pi_k(\boldsymbol{\tau})}{\partial\tau_j}-\sum_{k=1}^{m}\frac{\partial p_k}{\partial\tau_j}d_k(\mathbf{p}(\boldsymbol{\tau}))+n_j^* q_j^*(\boldsymbol{\tau}). \quad \text{(B2)}$$

Substituting definition $m_k=n_k^* q_k^*$ into expression (B2) yields

$$-\sum_{k=1}^{m}\tau_k\frac{\partial m_k(\boldsymbol{\tau})}{\partial\tau_j}=\sum_{k=1}^{m}\frac{(I_k+a)}{(a+\alpha_L)}n_k\frac{\partial\pi_k(\boldsymbol{\tau})}{\partial\tau_j}-\sum_{k=1}^{m}\frac{\partial p_k}{\partial\tau_j}d_k(\mathbf{p}(\boldsymbol{\tau}))+n_j^* q_j^*(\boldsymbol{\tau}),$$

or, equivalently, in a matrix form

$$-\boldsymbol{\tau}\left[\left(\partial m_k(\boldsymbol{\tau})/\partial\tau_j\right)_{k,j}\right]=B\left[\left(\sum_{k=1}^{m}\frac{(I_k+a)}{(a+\alpha_L)}n_k\frac{\partial\pi_k(\boldsymbol{\tau})}{\partial\tau_j}-\sum_{k=1}^{m}\frac{\partial p_k}{\partial\tau_j}d_k(\mathbf{p}(\boldsymbol{\tau}))+n_j^* q_j^*(\boldsymbol{\tau})\right)_{k,j=1,...,n}\right].$$

Multiplying both sides with $H = -\left[\left(\partial m_k(\boldsymbol{\tau})/\partial \tau_j\right)_{k,j}\right]^{-1}$ yields system (6).

B.2 Comment to the proof of Lemma (A.1)

This comment explains in more detail some derivations necessary to prove Lemma (A.1). Specifically, it shows how to calculate the determinant of matrix $\boldsymbol{R}$ (to show that it is invertible), and how to sign the elements of matrices $\boldsymbol{S} = \boldsymbol{R}^{-1}\boldsymbol{n}^*$, $\boldsymbol{S}^* = -\boldsymbol{R}^{-1}\boldsymbol{n}$, and $\boldsymbol{n}^{-1}\boldsymbol{R}$.

In order to calculate the determinant of matrix $\boldsymbol{R}$, I first apply a linear transformation to the rows of $\boldsymbol{R}$ (which does not change the determinant) and then use expansion by co-factors. More specifically,

$$\begin{aligned}
\det \boldsymbol{R} &= \det\begin{pmatrix} 1\text{+N}_1 & \sigma\text{N}_2 & ... & \sigma\text{N}_m \\ \sigma\text{N}_1 & 1\text{+N}_2 & ... & \sigma\text{N}_m \\ ... & ... & ... & ... \\ \sigma\text{N}_1 & \sigma\text{N}_2 & ... & 1\text{+N}_m \end{pmatrix} = \det\begin{pmatrix} 1\text{+N}_1 & \sigma\text{N}_2 & ... & \sigma\text{N}_m \\ \sigma\text{N}_1\text{-1-N}_1 & 1\text{+N}_2\text{-}\sigma\text{N}_2 & ... & 0 \\ ... & ... & ... & ... \\ \sigma\text{N}_1\text{-1-N}_1 & 0 & ... & 1\text{+N}_m \end{pmatrix} \\
&= (1\text{+N}_1)\det\begin{pmatrix} 1\text{+N}_2\text{-}\sigma\text{N}_2 & 0 & ... & 0 \\ 0 & 1\text{+N}_3\text{-}\sigma\text{N}_3 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1\text{+N}_m\text{-}\sigma\text{N}_m \end{pmatrix} - \\
&\quad -\sigma\text{N}_2\det\begin{pmatrix} \sigma\text{N}_1\text{-1-N}_1 & 0 & ... & 0 \\ \sigma\text{N}_1\text{-1-N}_1 & 1\text{+N}_3\text{-}\sigma\text{N}_3 & ... & 0 \\ ... & ... & ... & ... \\ \sigma\text{N}_1\text{-1-N}_1 & 0 & ... & 1\text{+N}_m\text{-}\sigma\text{N}_m \end{pmatrix} + \\
&\quad +\sigma\text{N}_3\det\begin{pmatrix} \sigma\text{N}_1\text{-1-N}_1 & 1\text{+N}_2\text{-}\sigma\text{N}_2 & ... & 0 \\ \sigma\text{N}_1\text{-1-N}_1 & 0 & ... & 0 \\ ... & ... & ... & ... \\ \sigma\text{N}_1\text{-1-N}_1 & 0 & ... & 1\text{+N}_m\text{-}\sigma\text{N}_m \end{pmatrix} - ... = \\
&= (1+N_1)\prod_{i=2}^{m}\left(1+(1-\sigma)N_i\right) - \sigma\text{N}_2\left(\sigma\text{N}_1\text{-1-N}_1\right)\prod_{j=3}^{m}\left(1+(1-\sigma)N_j\right) + \\
&\quad +\sigma N_3\left(-\left(\sigma\text{N}_1\text{-1-N}_1\right)\left(1\text{+N}_2\text{-}\sigma\text{N}_2\right)\prod_{j=4}^{m}\left(1+(1-\sigma)N_j\right)\right) - ... \\
&= (1+N_1)\prod_{i=2}^{m}\left(1+(1-\sigma)N_i\right) + \sigma\sum_{i=2}^{m}N_i\prod_{j=1,j\neq i}^{m}\left(1+(1-\sigma)N_j\right) > 0.
\end{aligned}$$

To sign the elements of matrix $\boldsymbol{S} = \boldsymbol{R}^{-1}\boldsymbol{n}^*$, I first derive the elements of the inverse matrix $\boldsymbol{R}^{-1}$ by using the well-known formula

$$\boldsymbol{R}^{-1} = \frac{1}{\det \boldsymbol{R}} (\text{cofactor matrix of } \boldsymbol{R})^T .$$

The same method (i.e., linear transformation of the rows followed by the expansion by co-factors) is used to calculate the transpose of the cofactor matric of $\boldsymbol{R}$. For example, the (2, 1)-element of the transpose of cofactor matrix of $\boldsymbol{R}$ is calculated as

$$\begin{aligned}
\left((\text{cofactor matrix of } \boldsymbol{R})^T\right)_{2,1} &= -\det \begin{pmatrix} \sigma \mathrm{N}_1 & \sigma \mathrm{N}_3 & ... & \sigma \mathrm{N}_m \\ \sigma \mathrm{N}_1 & 1\text{+N}_3 & ... & \sigma \mathrm{N}_m \\ ... & ... & ... & ... \\ \sigma \mathrm{N}_1 & \sigma \mathrm{N}_3 & ... & 1\text{+N}_m \end{pmatrix} \\
&= -\det \begin{pmatrix} \sigma \mathrm{N}_1 & \sigma \mathrm{N}_3 & ... & \sigma \mathrm{N}_m \\ 0 & 1\text{+N}_3\text{-}\sigma \mathrm{N}_3 & ... & \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1\text{+N}_m\text{-}\sigma \mathrm{N}_m \end{pmatrix} \\
&= -\sigma \mathrm{N}_3 \prod_{j=3}^{m} \left(1 + (1-\sigma)\, N_j\right) .
\end{aligned}$$

In turn, the (2, 2)-element of the transpose of cofactor matrix of $\boldsymbol{R}$ is calculated as

$$\begin{aligned}
\left((\text{cofactor matrix of } \boldsymbol{R})^T\right)_{2,2} &= \det \begin{pmatrix} 1\text{+N}_1 & \sigma \mathrm{N}_3 & ... & \sigma \mathrm{N}_m \\ \sigma \mathrm{N}_1 & 1\text{+N}_3 & ... & \sigma \mathrm{N}_m \\ ... & ... & ... & ... \\ \sigma \mathrm{N}_1 & \sigma \mathrm{N}_3 & ... & 1\text{+N}_m \end{pmatrix} \\
&= \det \begin{pmatrix} 1\text{+N}_1 & \sigma \mathrm{N}_3 & ... & \sigma \mathrm{N}_m \\ \sigma \mathrm{N}_1\text{-1-N}_1 & 1\text{+N}_3\text{-}\sigma \mathrm{N}_3 & ... & 0 \\ ... & ... & ... & ... \\ \sigma \mathrm{N}_1\text{-1-N}_1 & 0 & ... & 1\text{+N}_m \end{pmatrix} \\
&= (1+N_1) \prod_{i=3}^{m} (1 + (1-\sigma)\, N_i) + \sigma \sum_{i=3}^{m} N_i \prod_{j=1, j\neq i, 2}^{m} \left(1 + (1-\sigma)\, N_j\right) > 0.
\end{aligned}$$

and all other elements $(2, j)$, $j = 3..., m$, are calculated in the same way. This yields the formula for the $(2,2)$-th diagonal element of $\boldsymbol{S}$

$$\boldsymbol{S}_{22} = \frac{1}{\det \boldsymbol{R}}\left(\left((1+N_1)\prod_{i=3}^{m}(1+(1-\sigma)N_i) + \sum_{i=3}^{m}\sigma N_i \prod_{j=1,j\neq i,2}^{m}\left(1+(1-\sigma)N_j\right)\right)n_2^* \right.$$

$$\left. - \sum_{i=1,i\neq 2}^{m}\left(\sigma N_i \prod_{j=1,j\neq i,2}^{m}\left(1+(1-\sigma)N_j\right)\right)\sigma n_2^*\right) = \frac{n_2^*\left[(1+N_1)-\sigma^2 N_1\right]}{\det \boldsymbol{R}}\prod_{i=3}^{m}(1+(1-\sigma)N_i) > 0.$$

The formula for the (k,k)-th diagonal element of $\boldsymbol{S}$ is obtained in exact same way.

Further, the same reasoning yields the rest of the results needed to prove part (a) of Lemma (A.1). Specifically, the off-diagonal elements of matrix $\boldsymbol{S}$ are nonnegative as

$$\begin{aligned}
\boldsymbol{S}_{kl} &= \frac{1}{\det \boldsymbol{R}}\left(\left((1+N_1)\prod_{i=2,i\neq k}^{m}(1+(1-\sigma)N_i) + \sigma\sum_{i=2,i\neq k}^{m} N_i \prod_{j=1,j\neq i,k}^{m}\left(1+(1-\sigma)N_j\right)\right)\sigma n_l^* \right.\\
&\quad \left. - \sum_{i=1,i\neq k,l}^{m}\left(\sigma N_i \prod_{j=1,j\neq i,k}^{m}\left(1+(1-\sigma)N_j\right)\right)\sigma n_l^* - \left(\sigma N_l \prod_{j=1,j\neq l,k}^{m}\left(1+(1-\sigma)N_j\right)\right)n_l^*\right)\\
&= \frac{1}{\det \boldsymbol{R}}\sigma n_l\,(1+(1-\sigma)N_1)\prod_{j=2,j\neq l,k}^{m}\left(1+(1-\sigma)N_j\right) \geq 0.
\end{aligned}$$

Similarly, the partial derivatives $\partial q_k^*/\partial \tau_j$ are given by the respective elements of the matrix $\boldsymbol{S}^* = -\boldsymbol{R}^{-1}\boldsymbol{n}$, and the diagonal elements of $\boldsymbol{S}^*$ are negative

$$\begin{aligned}
\boldsymbol{S}_{kk}^* &= \frac{(-1)}{\det \boldsymbol{R}}\left(\left((1+N_1)\prod_{i=2,i\neq k}^{m}(1+(1-\sigma)N_i) + \sigma\sum_{i=2,i\neq k}^{m} N_i \prod_{j=1,j\neq i,k}^{m}\left(1+(1-\sigma)N_j\right)\right)(n_k+1) \right.\\
&\quad \left. - \sum_{i=1,i\neq k}^{m}\left(\sigma N_i \prod_{j=1,j\neq i,k}^{m}\left(1+(1-\sigma)N_j\right)\right)\sigma n_k\right)\\
&\leq -\frac{1}{\det \boldsymbol{R}}\prod_{i=2,i\neq k}^{m}(1+(1-\sigma)N_i)\left[(1+N_1)(n_k+1)-\sigma^2 N_1 n_k\right] < 0.
\end{aligned}$$

The off-diagonal elements of $\boldsymbol{S}^*$ are nonnegative

$$\begin{aligned}
\boldsymbol{S}_{kl}^* &= \frac{(-1)}{\det \boldsymbol{S}}\left(\left((1+N_1)\prod_{i=2,i\neq k}^{m}(1+(1-\sigma)N_i) + \sigma\sum_{i=2,i\neq k}^{m} N_i \prod_{j=1,j\neq i,k}^{m}\left(1+(1-\sigma)N_j\right)\right)\sigma n_l \right.\\
&\quad \left. - \sum_{i=1,i\neq k,l}^{m}\left(\sigma N_i \prod_{j=1,j\neq i,k}^{m}\left(1+(1-\sigma)N_j\right)\right)\sigma n_l - \left(\sigma N_l \prod_{j=1,j\neq k,l}^{m}\left(1+(1-\sigma)N_j\right)\right)(n_l+1)\right)\\
&= -\frac{1}{\det \boldsymbol{S}}(1+N_1-\sigma N_1)\,\sigma\,[n_l-N_l]\prod_{j=2,j\neq k,l}^{m}\left(1+(1-\sigma)N_j\right) \geq 0
\end{aligned}$$

which proves part (a) of Lemma (A.1).

To prove part (b) I need to show that the diagonal elements of matrix $\boldsymbol{n}^{-1}\boldsymbol{R}$ are positive, and the off-diagonal elements nonnegative. Again, using the exact same technique, I have

$$\begin{aligned}\left(\boldsymbol{n}^{-1}\boldsymbol{R}\right)_{kk} &= \frac{1}{\det \boldsymbol{n}}\left(\left((1+n_1)\prod_{i=2,i\neq k}^{m}(1+(1-\sigma)n_i)+\sigma\sum_{i=2,i\neq k}^{m}n_i\prod_{\substack{j=1,\\ j\neq i,k}}^{m}\left(1+(1-\sigma)n_j\right)\right)(N_k+1)\right.\\ &\quad \left.-\sum_{i=1,i\neq k}^{m}\left(\sigma n_i\prod_{\substack{j=1,\\ j\neq i,k}}^{m}\left(1+(1-\sigma)n_j\right)\right)\sigma N_k\right)\\ &\geq \frac{1}{\det \boldsymbol{n}}\prod_{i=2,i\neq k}^{m}(1+(1-\sigma)n_i)\left[(1+n_1)(N_k+1)-\sigma^2 n_1 N_k\right] > 0.\end{aligned}$$

In turn, he off-diagonal elements of $\boldsymbol{n}^{-1}\boldsymbol{R}$ are nonnegative as

$$\begin{aligned}\left(\boldsymbol{n}^{-1}\boldsymbol{R}\right)_{kl} &= \frac{(-1)}{\det \boldsymbol{n}}\left(\left((1+n_1)\prod_{i=2,i\neq k}^{m}(1+(1-\sigma)n_i)+\sigma\sum_{i=2,i\neq k}^{m}n_i\prod_{\substack{j=1,\\ j\neq i,k}}^{m}\left(1+(1-\sigma)n_j\right)\right)\sigma N_l\right.\\ &\quad \left.-\sum_{i=1,i\neq k,l}^{m}\left(\sigma n_i\prod_{\substack{j=1,\\ j\neq i,k}}^{m}\left(1+(1-\sigma)n_j\right)\right)\sigma N_l-\left(\sigma n_l\prod_{\substack{j=1,\\ j\neq k,l}}^{m}\left(1+(1-\sigma)n_j\right)\right)(N_l+1)\right)\\ &= \frac{1}{\det \boldsymbol{n}}(1+n_1-\sigma n_1)\,\sigma\,[N_l-n_l]\prod_{j=2,j\neq k,l}^{m}\left(1+(1-\sigma)n_j\right)\geq 0.\end{aligned}$$

B.3 Detailed proof of Proposition 1

To show that $\partial T_1(\sigma,\alpha,a)/\partial\sigma < 0$ for $\sigma \in 0,1$ and any (a,α), we proceed in 4 steps.

1. Describe the necessary condition for the interior solution for the tariff.

2. Represent T_1 as a ratio of two polynomials, $T_1(\sigma,\alpha,a)=\frac{N(\sigma,\alpha,a)}{D(\sigma,\alpha,a)}$. Show that $\frac{\partial T_1(\sigma,\alpha,a)}{\partial\sigma} < 0$ on $[0,1]$ iff $N_\sigma(\sigma,\alpha,a)/D_\sigma(\sigma,\alpha,a) > T_1(\sigma,\alpha,a)$ for all $\sigma \in 0,1$ and (a,α) delivering an interior solution.

3. Introduce an auxiliary function $A(\sigma,\alpha,a)$ and show that $N_\sigma(\sigma,\alpha,a)/D_\sigma(\sigma,\alpha,a) \geq A(\sigma,\alpha,a)$ for any admissible (σ,α,a).

4. Show that $A(\sigma,\alpha,a) \geq T_1(\sigma,\alpha,a)$ for any admissible (σ,α,a).

Steps 2, 3 and 4 imply that $\partial T_1(\sigma,\alpha,a)/\partial\sigma < 0$. The proof $\partial T_2(\sigma,\alpha,a)/\partial\sigma > 0$ is similar.

Step 1. Necessary condition on the tariff for the interior solution.

We focus on non-prohibitive tariffs (i.e., positive imports). From system (12) of Appendix A in the main text, a necessary condition for positive imports at $\sigma=0$ is

$$q^*(0,\tau_1,\tau_2)=\frac{(1-c)-2\tau_1(0,\alpha,a)}{3}\geq 0,$$

or, equivalently,

$$\frac{\tau_1(0,\alpha,a)}{(1-c)} \leq \frac{1}{2}$$

In turn, $\tau_1(0,\alpha,a)$, as given by equation (13) in the main text, is equal to

$$\tau_1(0,\alpha,a) = (1-c)\frac{(3a+\alpha+2)}{(9a+11\alpha-2)}$$

So, the necessary condition for the interior solution becomes

$$\begin{aligned} \frac{(3a+\alpha+2)}{(9a+11\alpha-2)} &\leq \frac{1}{2} \Leftrightarrow \\ a+3\alpha &\geq 2 \end{aligned} \tag{B3}$$

(condition (14). in the main text).

Step 2. Necessary and sufficient condition for $T_1(\sigma,\alpha,a)$ to decline in σ.

System (13) and definition (9) in the main text give the following expression for $T_1(\sigma,\alpha,a)$

$$T_1(\sigma,\alpha,a) = \frac{(\sigma+3)\left[4(a+1)(a+2\alpha)\sigma^3+4(\alpha^2-3a\alpha-2a-3a^2-3\alpha)\sigma^2-(9a^2+26a\alpha+10a+13\alpha^2+14\alpha)\sigma+(3a+\alpha+2)(9a+11\alpha)\right]}{4(a+2\alpha-1)(a+2\alpha)\sigma^4+(14a-118a\alpha-45a^2-81\alpha^2+22\alpha)\sigma^2+(9a+11\alpha-2)(9a+11\alpha)}$$

Denote the numerator of $T_1(\sigma,\alpha,a)$ by $N(\sigma,\alpha,a)$ and the denominator by $D(\sigma,\alpha,a)$.

Notice that $D(\sigma,a)$ decreases in σ for any admissible a. Indeed, $D(\sigma,\alpha,a)$ is a 4-th degree polynomial in σ

$$\begin{aligned} D(\sigma,\alpha,a) &= 4(a+2\alpha-1)(a+2\alpha)\sigma^4+(14a-118a\alpha-45a^2-81\alpha^2+22\alpha)\sigma^2 \\ &\quad +(9a+11\alpha-2)(9a+11\alpha). \end{aligned}$$

The first derivative of $D(\sigma,\alpha,a)$ is negative

$$\begin{aligned} \frac{\partial D(\sigma,\alpha,a)}{\partial\sigma} &= 16\sigma^3(a+2\alpha)(a+2\alpha-1)+2\sigma\left(14a-118a\alpha-45a^2-81\alpha^2+22\alpha\right) \\ &\leq 16\sigma(a+2\alpha)\underbrace{(a+2\alpha-1)}_{>0 \text{ by } (B3)}+2\sigma(14a-118a\alpha-45a^2-81\alpha^2+22\alpha) \\ &= -2\sigma(a+\alpha)(37a+49\alpha-6) \end{aligned}$$

As

$$37a+49\alpha-6 = 34a+40\alpha+3(a+3\alpha-2) > 0,$$

we conclude that $D_\sigma(\sigma,\alpha,a) < 0$. Further, note that

$$D(1,\alpha,a) = 8(a+\alpha)(5a+7\alpha-1) = 8(a+\alpha)(4a+4\alpha+1+(a+3\alpha-2)) > 0.$$

As $D(\sigma, \alpha, a)$ declines in σ on $0, 1$ and is positive at $\sigma = 1$, we conclude that $D(\sigma, \alpha, a) > 0$ for $\sigma \in [0, 1]$.

Since $\partial D(\sigma, \alpha, a)/\partial\sigma < 0$ and $D(\sigma, \alpha, a) > 0$,

$$\frac{\partial T_1(\sigma, \alpha, a)}{\partial \sigma} < 0 \Leftrightarrow$$
$$\frac{N_\sigma(\sigma, \alpha, a) D(\sigma, \alpha, a) - D_\sigma(\sigma, \alpha, a) N(\sigma, \alpha, a)}{D^2(\sigma, \alpha, a)} < 0 \Leftrightarrow$$
$$N_\sigma(\sigma, \alpha, a) D(\sigma, \alpha, a) < D_\sigma(\sigma, \alpha, a) N(\sigma, \alpha, a) \Leftrightarrow$$
$$\frac{N_\sigma(\sigma, \alpha, a)}{D_\sigma(\sigma, \alpha, a)} > \frac{N(\sigma, \alpha, a)}{D(\sigma, \alpha, a)}.$$

Step 3. Auxiliary function $A(\sigma, \alpha, a)$, **such that** $\frac{N_\sigma(\sigma,\alpha,a)}{D_\sigma(\sigma,\alpha,a)} \geq A(\sigma, \alpha, a)$.

Define $A(\sigma, \alpha, a)$ - a linear function of σ

$$A(a, \alpha, \sigma) = 3\frac{3a + \alpha + 2}{9a + 11\alpha - 2} - 8\frac{a + \alpha + 8a\alpha + 6\alpha^2 + 2}{(9a + 11\alpha - 2)(37a + 49\alpha - 6)}\sigma.$$

Consider

$$\Delta_1 = \frac{N_\sigma(\sigma, \alpha, a)}{D_\sigma(\sigma, \alpha, a)} - A(a, \alpha, \sigma) = \frac{4(\sigma - 1)}{(9a + 11\alpha - 2)(37a + 49\alpha - 6)} \frac{L(\sigma, \alpha, a)}{D_\sigma(\sigma, \alpha, a)}$$

where $L(\sigma, \alpha, a)$ is a cubic polynomial with respect to σ,

$$L(\sigma, \alpha, a) = \left(3a + 5\alpha + 9a\alpha + 7\alpha^2\right)(37a + 49\alpha - 6)(9a + 11\alpha - 2) + \sigma\,[(37a + 49\alpha - 6)$$
$$(2a + 6\alpha + 176a\alpha + 105\alpha^2 - 39\alpha^3 - 34a\alpha^2 + 9a^2\alpha + 63a^2)\big] + 8\sigma^2\,[(a + 2\alpha)(21a + 3\alpha$$
$$-17a\alpha - 9\alpha^2 + 10)(9a + 11\alpha - 2)\big] + 32\sigma^3\left[(a + 2\alpha)(a + 2\alpha - 1)\left(a + \alpha + 8a\alpha + 6\alpha^2 + 2\right)\right].$$

To sign Δ_1, note that $D_\sigma(\sigma, \alpha, a) \leq 0$ and $\sigma - 1 < 0$. Further, by (B3)

$$9a + 11\alpha - 2 = 8a + 8\alpha + (a + 3\alpha - 2) > 0$$

and

$$37a + 49\alpha - 6 = 34a + 40\alpha + 3(a + 3\alpha - 2) > 0.$$

Finally, one can show that the coefficients of the polynomial $L(\sigma, \alpha, a)$ are all positive. Indeed, the constant term is positive as

$$\left(3a + 5\alpha + 9a\alpha + 7\alpha^2\right)(37a + 49\alpha - 6)(9a + 11\alpha - 2) > 0.$$

The coefficient by σ is positive

$$(37a+49\alpha-6)\,(2a+6\alpha+176a\alpha+105\alpha^2-39\alpha^3-34a\alpha^2+9a^2\alpha+63a^2)\,>\,0$$

since

$$\begin{aligned} 2a+6\alpha+176a\alpha+105\alpha^2-39\alpha^3-34a\alpha^2+9a^2\alpha+63a^2 &\geq \\ 2a+6\alpha+176a\alpha+105\alpha^2-39\alpha^2-34a\alpha+9a^2\alpha+63a^2 &= \\ a+6\alpha+142a\alpha+66\alpha^2+9a^2\alpha+63a^2 &> \quad 0. \end{aligned}$$

The coefficient by σ^2 is also positive,

$$8\,(a+2\alpha)\left(21a+3\alpha-17a\alpha-9\alpha^2+10\right)(9a+11\alpha-2)\,>\,0,$$

since

$$21a+3\alpha-17a\alpha-9\alpha^2+10\,>\,21a+3\alpha-17a-9+10=4a+3\alpha+1\,>\,0.$$

And, finally, the coefficient by σ^3 is also positive by (B3)

$$32\sigma^3\,(a+2\alpha)\,(a+2\alpha-1)\left(a+\alpha+8a\alpha+6\alpha^2+2\right)\,>\,0.$$

Thus, for any $\sigma\,\in 0, 1,\ \Delta_1\geq 0$.

Step 4. Proof of $A\,(\sigma,\alpha,a)\geq T_1\,(\sigma,\alpha,a)$.

Consider the difference Δ_2 between $T_1\,(\sigma,\alpha,a)$ and $A(a,\alpha,\sigma)$

$$\Delta_2\,=T_1\,(\sigma,\alpha,a)-A(a,\alpha,\sigma)=\frac{4\sigma M\,(\sigma,\alpha,a)}{(9a+11\alpha-2)\,(37a+49\alpha-6)\,D\,(\sigma,\alpha,a)}$$

where $M\,(\sigma,a)$ is a 4th degree polynomial with respect to σ,

$$\begin{aligned} M\,(\sigma,\alpha,a) &= (9a+11\alpha-2)\,(54a+74\alpha-238a\alpha-181\alpha^2-211\alpha^3-416a\alpha^2-189a^2\alpha-93a^2) \\ &-2\sigma\left[(37a+49\alpha-6)\left(2a+4\alpha+29a\alpha+16\alpha^2-29\alpha^3-49a\alpha^2-18a^2\alpha+9a^2\right)\right] \\ &+\sigma^2\left[-433\alpha^4-\alpha^3\,(247a-1555)+\alpha^2\left(449a^2+2585a-760\right)\right. \\ &\left.+\alpha\left(279a^3+1361a^2-912a+124\right)+243a^3-280a^2+68a\right] \\ &+2\sigma^3\,(a+2\alpha)\left[(1-\alpha)\,(37a+49\alpha-6)\,(5a+3\alpha+2)+4\sigma\,(a+2\alpha-1)\left(a+\alpha+8a\alpha+6\alpha^2+2\right)\right]. \end{aligned}$$

To sign Δ_2, notice that, as shown above, $(9a+11\alpha-2)\,(37a+49\alpha-6)\,D\,(\sigma,\alpha,a)\,>\,0$. Consider $M\,(\sigma,\alpha,a)$. Its second derivative $\partial^2 M\,(\sigma,\alpha,a)\,/\partial\sigma^2$ is a convex quadratic parabola with the minimum at some $\sigma < 0$. Indeed,

$$\begin{aligned}\frac{\partial^2 M(\sigma,\alpha,a)}{\partial\sigma^2} &= 2(68a+124\alpha+449a^2\alpha^2-912a\alpha-760\alpha^2+1555\alpha^3-433\alpha^4+\\ &\quad 2585a\alpha^2+1361a^2\alpha-247a\alpha^3+279a^3\alpha-280a^2+243a^3)+\\ &\quad 12\sigma(1-\alpha)(37a+49\alpha-6)(5a+3\alpha+2)(a+2\alpha)+\\ &\quad 96\sigma^2(a+2\alpha-1)\left(a+\alpha+8a\alpha+6\alpha^2+2\right)(a+2\alpha).\end{aligned}$$

and its minimum is at

$$\sigma_{\min} = -\frac{(1-\alpha)(37a+49\alpha-6)(5a+3\alpha+2)}{16(a+2\alpha-1)\left(a+\alpha+8a\alpha+6\alpha^2+2\right)} < 0.$$

Thus, it is increasing at $\sigma \in 0, 1$. It is positive at $\sigma = 0$

$$\begin{aligned}\frac{\partial^2 M(0,\alpha,a)}{\partial\sigma^2} &= 2(68a+124\alpha+449a^2\alpha^2-912a\alpha-760\alpha^2+1555\alpha^3-433\alpha^4\\ &\quad +2585a\alpha^2+1361a^2\alpha-247a\alpha^3+279a^3\alpha-280a^2+243a^3)\\ &= 2(140a^2+456a\alpha+380\alpha^2)(a+3\alpha-2)+68a+124\alpha+449a^2\alpha^2\\ &\quad +415\alpha^3-433\alpha^4+837a\alpha^2+485a^2\alpha-247a\alpha^3+279a^3\alpha+103a^3\\ &> 268a+124\alpha+449a^2\alpha^2\\ &\quad +415\alpha^3-433\alpha^4+837a\alpha^2+485a^2\alpha-247a\alpha^3+279a^3\alpha+103a^3\\ &> 2\left(124\alpha+415\alpha^3-433\alpha^4+837a\alpha^2-247a\alpha^3\right)\\ &> 2\left(124\alpha^4+415\alpha^4-433\alpha^4+837a\alpha^2-247a\alpha^2\right)\\ &= 4\alpha^2\left(295a+53\alpha^2\right) > 0,\end{aligned}$$

So we conclude that for any $\sigma \in 0, 1$,

$$\frac{\partial^2 M(\sigma,\alpha,a)}{\partial\sigma^2} > 0.$$

As a result, $M(\sigma,\alpha,a)$ is a convex function of σ at $0, 1$ for any (α, a) and reaches its maximum at (either of) the corner points of the segment. But $M(\sigma, a) < 0$ both at $\sigma = 0$ and $\sigma = 1$. Indeed,

$$\begin{aligned}M(0,\alpha,a) = (9a+11\alpha-2)\,54a+74\alpha-238a\alpha\\ -181\alpha^2-211\alpha^3-416a\alpha^2-189a^2\alpha-93a^2 < 0,\end{aligned}$$

as

$$54a+74\alpha-238a\alpha-181\alpha^2-211\alpha^3-416a\alpha^2-189a^2\alpha-93a^2=$$

$$(27a+37\alpha)(2-3\alpha-a)-\left(120a\alpha+70\alpha^2+211\alpha^3+416a\alpha^2+189a^2\alpha+66a^2\right)<0.$$

Moreover,

$$M(1,\alpha,a)=-2(a+\alpha)(9a+11\alpha-2)\left(49a+81\alpha+22a\alpha+14\alpha^2-14\right)<0,$$

as

$$49a+81\alpha+22a\alpha+14\alpha^2-14=7(a+3\alpha-2)+42a+60\alpha+22a\alpha+14\alpha^2>0.$$

So we conclude that $M(\sigma,\alpha,a)\leq 0$, which implies $\Delta_2\leq 0$.